Repunits and Harmony

Evgenii E. Meerson

1. Introduction

During the last 10-20 years attention in the several Internet sites attracted to repunits, the symmetric integers that have all digits in decimal system (base**10**) equal to unity [1-2]. In the present article word repunit is used for more general class of integers. We shall imply that repunit is the integer that has all digits equal to unity in the given positional system of numeration non obligatory decimal, for example tridecimal system (base **13**) or sexageasimal system (base **60**). In the second section of the article we show that some repunits are connected by simple mathematical relation with other symmetrical integers that have symmetry of natural series, which have digital presentation in the form **123456…**. In the third section presented prime factorization of some of those repunits. In the fourth section of the article discussed possible relation of repunits to particle physics.

Now we shall say some words about the name of the article. Natural series is $\mathbf{1^{+1}, 2^{+1}, 3^{+1}, 4^{+1}, \ldots}$ Harmonic series is $\mathbf{1^{-1}, 2^{-1}, 3^{-1}, 4^{-1}, \ldots}$ We can see that both serieses have the same digital construction and only difference is in the sign of the powers. The author devotes his modest article to the memory of pianist and compositor W. A. Mozart which is 250 in this year (2006), and which had been one of the greatest masters of the harmony. W. A. Mozart had been died in December 5. The author of the article was born also in December 5, and will 60 in this year. This mystical coincidence forces him to show preference to word "Harmony" but not "Nature".

2. Minimum error repunit

First at all we shall introduce some classification of numbers that will be useful for us further. We define a rank of a positional number system as the number of figures this system uses. E.g., the rank of the decimal system is equal to nine. For **N**- rank system of numeration we define five types of numbers. 1) We say that number is filled number if zero is not used for number writing. In contrary case number will be named unfilled. E.g., **1215** is filled number and **1205** is unfilled number. 2) We say that number is one-digit if only one digit is used for number writing. In contrary case number will be named multidigital number. E.g., **0.03** is one-digit number and **0.033** or **0.034** are multidigital numbers. 3) Filled one-digit integer we name digit. E.g., **2** is digit but **20** isn't digit, so as it one-digit integer but not filled, 4) We say that integer is one-digit prime if the integer simultaneously digit and prime. E.g., in the tridecimal system **10** isn't one-digit prime so as it one-digit integer and prime but not digit. 5) We say that integer is unfilled repunit if in addition to unity zero is also used for the number writing. E.g., **1011** is unfilled repunit. Further we shall compare transformation characteristics of number during permutation of digits. In this case we suppose that all position occupied by zeros unchanged. In other words only those permutation are in the consideration in which every position occupied by zero befor the permutation must always be occupied by zero after the permutation. Taking this in mind we can say that one digit number and repunit unchanged after arbitrary permutation of digits.

If we will compare repunits with the digit we will find some common properties. First is that as the digit repunits may be prime and composed integer [1]. Second is that repunits unchanged after arbitrary permutations of digits. But there is also some differences. First is connected with the dual property of digit (and also every one-digit number), that analogous to dual property of particle in the quantum mechanic (particle is simultaneously particle and wave). In the case of one-digit number it may be say that all its digits different so as only one digit used for writing and for this reason no two equal digits used for writing the integer. If we accept this version we can find between multidigital integers totality of filled integers with the same properties see also [3]. But with equal success we can also say that in one digit number all digits are equal so as no two different digits used for the integer writing. In this case repunits is the totality of multidigital integers that have the same properties. Second difference is, that digit is composed if and only if sum of it's digits is composed. In the case of repunits if sum of digit composed repunit is always composed [1] (further we shall prove that this propery is true for arbitrary system of numeration), and may be case that sum of repunit digits is prime but repunit itself composed [1-2].

Let us consider a number system of **N** rank and within it an **N**-digit repunit $\mathbf{R_N}$ (digit **1**, or unity, exists in any positional number system). The following Theorem 1 will be proved: "For any numeration system of **N** rank, the following expression is true:

$$\mathbf{R_N \equiv 1\,1\,1\,1\ldots1 \equiv {}_{(N)}1\,1\,1\,1\ldots1_{(N)} = {}_{(N)}1\,2\,3\,\ldots\,a_{N-3}\,a_{N-2}\,a_N \cdot a_N \equiv b_N \cdot a_N}\text{ "} \qquad (1)$$

We use here generalization of notations introduced in [1] for description of repunit. Subscript in brackets of in the right corner indicate number of digits, subscript in the brackets of left corner indicate rank of the system numeration which is used for integer presentation. In the expression, the number $\mathbf{b_N}$ is composed of **N-1** unrepeated figures, placed in the order of increase concerning the numbers they represent, from left to right. The only figure that remains unutilized in the representation (composition) of the number $\mathbf{b_N}$ is the figure $\mathbf{a_{N-1}}$, as it is omitted in (1). **1, 2, 3, …,** $\mathbf{a_N}$ are the figures corresponding to the natural series, $\mathbf{a_N}$ being the last digit corresponding to the largest one-digit number of the system under consideration.

The Theorem is proved by way of induction:

According to (1), the last position in the representation of number $\mathbf{b_N \cdot a_N}$ is occupied by the last digit of the number

$$\mathbf{a_N \cdot a_N \equiv a_N \cdot ({}_{(N)}10\text{-}1) = a_N0 - a_N = {}_{(N)}a_{N-1}1} \qquad (2)$$

Hence, according to (2), the last position in the representation of the number $\mathbf{b_N \cdot a_N}$ is, indeed, 1 (unity).

While inferring (2), we used the fact that it follows from the definition of a positional number system of N rank

$$\mathbf{1 + a_N \equiv {}_{(N)}10}$$

$$\mathbf{a_N \cdot {}_{(N)}10 \equiv a_N0 = a_{N-1}0 + {}_{(N)}10} \qquad (3)$$

The next to last position in the number $\mathbf{b_N \cdot a_N}$ is occupied, according to (1-2), by the last digit of the expression

$$\mathbf{a_{N-2} \cdot a_N + a_{N-1} = a_{N-2}0 - a_{N-2} + a_{N-1} = a_{N-2}0 + 1 = {}_{(N)}a_{N-2}1} \qquad (4)$$

Is follows from (4) that the next to last position in the number $\mathbf{b_N \cdot a_N}$ is also occupied by **1** (unity). Similarly, **1** stands in the next to next the last position (i.e., the third position from the right) in the number $\mathbf{b_N \cdot a_N}$. Indeed, the next to next the last position in the number $\mathbf{b_N \cdot a_N}$ is occupied, according to (1,4), by the last digit of the expression

$$\mathbf{a_{N-3} \cdot a_N + a_{N-2} = a_{N-3} \cdot ({}_{(N)}10\text{-}1) + a_{N-2} = {}_{(N)}a_{N-3}1} \qquad (5)$$

We have shown, therefore, that at least three last digits in the number $\mathbf{b_N \cdot a_N}$ are equal to **1**.

Suppose now that the digit $\mathbf{a_t}$ (occupying $\mathbf{t}^{\underline{th}}$ position in the number $\mathbf{b_N}$) is multiplied by the number $\mathbf{a_N}$ and to the product thus obtained we add the first digit of the number $\mathbf{{}_{(N)}a_{t+1}\, a_{t+2} \ldots a_{N-3}\, a_{N-2}\, a_N \cdot a_N}$. Let as suppose that as a result we shall obtain the number $\mathbf{{}_{(N)}a_t1}$ (in full compliance with (4-5)), which means that in the number $\mathbf{b_N \cdot a_N}$, the $\mathbf{(N\text{-}t)}^{\underline{th}}$ position from the right is occupied by **1**.

Under this assumption let us now find the last digit in the number which is formed by multiplication of the digit $\mathbf{a_{t-1}}$, occupying $\mathbf{(t\text{-}1)}^{\underline{th}}$ position in the number $\mathbf{b_N}$, by $\mathbf{a_N}$, followed by adding $\mathbf{a_t}$ to the product thus obtained

$$\mathbf{a_{t-1} \cdot a_N + a_t = a_{t-1} \cdot ({}_{(N)}10\text{-}1) + a_t = {}_{(N)}a_{t-1}1} \qquad (6)$$

We see from (6), that **1** also occupies $\mathbf{(N + 1 - t)}^{\underline{th}}$ position from the right in the number $\mathbf{b_N \cdot a_N}$. Theorem 1 is proved.

It follows from Theorem 1 that

For the repunit $\mathbf{R_N}$ always exist an one-digit divisor different from **1**.

In the factorization of the repunit $\mathbf{R_N}$ into the factors $\mathbf{b_N}$ and $\mathbf{a_N}$, there are only two selected figures $\mathbf{a_{N-1}}$ and $\mathbf{a_N}$ that correspond to the numbers **N и N-1.** The figure $\mathbf{a_N}$ is presented twice, while the figure $\mathbf{a_{N-1}}$ is altogether absent. All the rest of the figures of the system that correspond to first **N-2** integers of natural series (i.e., **N-2** digits of the $\mathbf{b_N}$) appear just once in (1).

Following the method described above it easy to show that for **m**-digit integer $\mathbf{{}_{(N)}h_{(m)}}$ that has in the system of rank **N** representation

$$\mathbf{{}_{(N)}H_{(m)} \equiv {}_{(N)}1234\ldots m} \qquad (7)$$

where **1234…m** is natural series finished by **m**, following mathematical relation is true

$$\mathbf{{}_{(N)}H_{(m)} \cdot a_N = {}_{(N)}111\ldots1_{(m-1)} \cdot {}_{(N)}100 + (N+1-m) \equiv {}_{(N)}R_{(m-1)} \cdot {}_{(N)}100 + (N+1-m)} \qquad (8)$$

Integer $\mathbf{{}_{(N)}H_{(m)}}$ we shall name "harmonic integer". Maximum harmonic integer in the system of numeration of rank **N** obviously is $\mathbf{{}_{(N)}H_{(N)} \equiv H_N}$.

We can rewrite relation (8) in the form more like (1)

$$\mathbf{{}_{(N)}R_{(m-1)} = {}_{(N)}H_{(m)} \cdot N \cdot ({}_{(N)}100)^{-1} - (N+1-m) \cdot ({}_{(N)}100)^{-1}} \qquad (9)$$

It is convenient for further application to rewrite (1) in the form

$$\mathbf{R_N = {}_{(N)}H_{(N-1)} \cdot N + N} \qquad (10)$$

Relation (9) and(10) very similar. In the both cases repunits is in the left parts of equations and harmonic integers multiplied by rational factors plus additional rational reminders are in the right part of equations. If we neglect by rational reminders (that is we suppose that the repunit is divider of the harmonic integer) we shall get in the case of (9) relative error δ_m

$$\delta_m = {}_{(N)}R_{(m-1)}^{-1} \cdot (N+1-m) \cdot (N+1)^{-2} \approx (N+1-m) \cdot (N+1)^{-(m+1)} \quad (11)$$

In the case of equation (10) we shall get relative error δ

$$\delta = N/(R_N) \approx (N+1)^{-(N-1)} \quad (12)$$

It follows from (11-12) that d is minimum in the case of (11) with m=N. In this case

$$\delta \approx (N+1)^{-(N+1)} \quad (13)$$

This means that among all repunits with the numbers of digits less or equal to **N** where **N** is rank of system of numeration the repunit ${}_{(N)}R_{(N-1)}$ with the best relative approximation is the product maximum harmonic integer H_N and one digit number ${}_{(N)}0.0N$.

$$ {}_{(N)}R_{(N-1)} \approx {}_{(N)}0.0N \cdot H_N \quad (14)$$

For this reason we shall name this repunit minimum error repunit. All numbers in (14) have specific symmetry. Number in left part and first factor in right part is invariant relative digits permutations. Second factor in right part (harmonic integer) has natural series symmetry maximum available for digits of given system of numeration. Relation (14) is approximate with mathematical point of view, but from physical point of views in the most important cases (14) must be considered as exact. Really, In the case of decimal system for example (**N=9**) relative error in (14) is $\sim 10^{-10} = 10^{-8}$ **%** in accordance with (13). Most of physical fundamental constants are measured today with much worse accuracy [4] and may will never be measured with the relative accuracy 10^{-10}.

3. Test № 2

Let us consider repunit ${}_{(N)}R_{(M)}$. Following theorem (factorization theorem) is true. " repunit ${}_{(N)}R_{(M)}$ may be represented as product of two integers repunit and unfilled repunit if M composed"

Proof. Let us suppose that **M=P·Q** where **P** and **Q** are integers. Then we can obviously write

$$ {}_{(N)}R_{(M)} = {}_{(N)}R_{(P)} \cdot {}_{(N)}10^{(M-P)} + {}_{(N)}R_{(P)} \cdot {}_{(N)}10^{(M-2P)} + \ldots + {}_{(N)}R_{(P)} \cdot {}_{(N)}10^{(M-(Q-1)\cdot P)} + {}_{(N)}R_{(P)} =$$

$$ = {}_{(N)}R_{(P)} \cdot [{}_{(N)}10^{(M-P)} + {}_{(N)}10^{(M-2P)} \ldots + {}_{(N)}10^{(M-(Q-1)\cdot P)} + 1] \quad (15)$$

Summing expression in the brackets we obtain

$$ {}_{(N)}R_{(M)} = {}_{(N)}R_{(P)} \cdot {}_{(N)}(1000\ldots01000\ldots01000\ldots0\ldots1000\ldots01) \quad (16)$$

Here factor in brackets (16) is integer composed of **Q** units with **P-1** zeros between the every neighbor units. Theorem is proven.

We shall say that prime C_N is considered as conforming to test №2 if it devisor of minimum error repunit ${}_{(N)}R_{(N-1)}$. This is definition of test №2. For definition of teast №1 see [3]. Using above theorem it is easily to find totalities of C_N for **N** lying in the region $6 \le N \le 11$.

$$\begin{aligned}
{}_{(6)}R_{(5)} &= 2801 \\
{}_{(7)}R_{(6)} &= 3\cdot3\cdot3\cdot19\cdot73 \\
{}_{(8)}R_{(7)} &= 547\cdot1093 \\
{}_{(9)}R_{(8)} &= 11\cdot73\cdot101\cdot137 \\
{}_{(10)}R_{(9)} &= 7\cdot19\cdot1772893 \\
{}_{(11)}R_{(10)} &= 13\cdot19142\cdot22621
\end{aligned} \quad (17)$$

We can see from (17) that 1) there is no one prime common to two totalities, 2) only two totalities have common number of primes, 3) Number m of primes in the 6 totalities lies in the region $1 \leq m \leq 5$, 4) not always all primes in the totality are different.

Let us consider totality in (17) that corresponds to decimal system. We notice three facts. 1) $_{(9)}\mathbf{H}_{(8)}$ is $_{(N)}\mathbf{R}_{(8)}$, 2) The totality consist of four primes, 3) totality include prime **137**. If we will do factorization repunits $_{(N)}\mathbf{R}_{(8)}$ we shall find that next two repunits that include factor **137** is $_{(126)}\mathbf{R}_{(8)}$ and $_{(135)}\mathbf{R}_{(8)}$. Second case trivially follows from the above factorization theorem so as $\mathbf{137} = {}_{(135)}\mathbf{R}_{(2)}$. We can conclude that probability that the facts (1-3) will be fulfilled for one integer in the one of positional system of numeration is very low.

4. Possible application to particle physics

Let us consider totality of **M** different boson quantum fields $\mathbf{B_1, B_2, B_3, \ldots B_M}$, that are distinguished by some discrete parameter η. Let us suppose that probability of distribution of quantums of every of fields is uniform in the configurational space (for simplicity in the coordinate space). Let us also suppose that all the quantums have equal impulses spins and other measurable dynamic characteristics. Filled **M** digital integer $_{(N)}\mathbf{A}_{(M)}$ may describe such state of the fields if we will suppose that first digit corresponds to number of quantum field $\mathbf{B_1}$, second to number of quantum of $\mathbf{B_2}$, third to number quantum of $\mathbf{B_3}$ and so all. Of course it is supposed that number of quantums of every field doesn't exceed the **N** - rank system of numeration. Non filled integer with **K** zeros and **M-K** digits corresponds to state of fields when **K** of **M** fields have zero of quantums in the space. If the number of quantum of every field is unity then $_{(N)}\mathbf{A}_{(M)} = {}_{(N)}\mathbf{R}_{(M)}$.

In particle physics strong interaction realized by **8** neutral equal boson fields that are differed only by their color characteristics so the η may be color parameter in this case. Weak interaction realized by two boson fields one electrically charged and second neutral so in this case η may be the charge of quantum, and electromagnetic interaction realized by one neutral boson field [5]. Every interaction characterized by their dimensionless constants of interaction. At low energy limit there is one constant for strong interaction $\alpha_s^{-1} \approx \mathbf{1}$, two constants for weak interaction $\alpha_w^{-1} \approx \mathbf{29.01}$ and $\alpha_z^{-1} \approx \mathbf{23.10}$, one constant for electromagnetic $\alpha^{-1} \approx \mathbf{137.0359895(61)}$ – where α is fine structure constant [5]. Nobody know why these constants have such values, no the theory or Principe predicted other values. So there are four dimensionless constants of interaction. As we remember minimum error repunit $_{(9)}\mathbf{R}_{(8)}$ also has four prime factors and one of them is prime **137,** that with the relative exactness better than 0.04% coincide with the α^{-1} . Is this accident coincidence? The $_{(9)}\mathbf{R}_{(8)}$ may describe such mentioned above uniform state of **8** gluon fields that realize strong interaction when every of the color fields has only one quantum in the universe. Is it possible to suppose that our universe was created so that at the beginning of the creation was fulfilled relations?

$$\alpha_s^{-1} = \mathbf{11}, \quad \alpha_z^{-1} = \mathbf{73}, \quad \alpha_w^{-1} = \mathbf{101}, \quad \alpha^{-1} = \mathbf{137} \tag{18}$$

And

$$\alpha_s^{-1} \cdot \alpha_z^{-1} \cdot \alpha_w^{-1} \cdot \alpha^{-1} = {}_{(9)}\mathbf{R}_{(8)} \approx \mathbf{0.09} \cdot \mathbf{H_9} \tag{19}$$

That may be named as "maximum harmony Principe". The base for this supposition may be the fact that precise experimental measurements show that α really slow changed with the time [6]. We don't know the same facts about the other constants, but accuracy of their measurements today is too bad to notice their changes.

In the previous section of the article we obtained integer **137** from symmetry characteristics of integers presented in the positional system of numeration, that follow from the pure mathematics, but it is known that in our time mathematics very quickly transformed to physics [7]

Let us retrace again our arguments. The existence of **8** color gluon fields with the one dimensionless constant of interaction follows from the **SU(3)** symmetry experimentally observed for the strong interaction [5]. Combining the existence of the **8** similar fields and symmetry characteristics of the integers we can come to decimal system in which repunit composed by **8** units has special symmetry property (minimum error repunit). After this suddenly we notice that this repunit is factorized by **4** primes and one of these primes with good accuracy coincides with one of dimensionless constants. After this we suddenly notice that complete number of dimensionless constants of interactions is also **4** and that values of all these constants are the same order as values of the **4** mentioned above primes. Existence of the **4** dimensionless constants follows from (additional to the **SU(3)** symmetry) experimentally observed **SU(2)** symmetry of weak interaction and **U(1)** symmetry of the electromagnetic interaction [5]. All this looks as the chain of accident coincidences in the detective story of the yang Scottish-Irish physician Arthur Conan Doyle [8]. But if we at the beginning will start from the decimal system of numeration we obtain more logically based process. Decimal system leads us to repunit composed by **8** units that is factorized by **4** primes. From this immediately follow **8** gluon fields and the **SU(3)**, **SU(2)** and **U(1)** symmetries with their **4** dimensionless constants of interaction.

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